Mathematics: analysis and approaches
Higher level
Paper 3

Name

Date: ____________________

1 hour

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

exam: 3 pages

Answer all questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 29]

In this question, you will investigate definite and indefinite integrals where the integrand is a rational function.

(a) (i) Show that $\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{4}$. [2]

(ii) Hence, write down the result for $\int_0^k \frac{1}{1+x^2}\,\mathrm{d}x$. [1]

(b) (i) Show that $\frac{1}{1-x^2} = \frac{1}{2(1-x)} + \frac{1}{2(1+x)}$. [5]

(ii) Hence, find $\int_0^{\frac{1}{2}} \frac{1}{1-x^2}\,\mathrm{d}x$. Give your answer in the form $a\ln b$ where $a, b \in \mathbb{R}$. [5]

(iii) Given that $\int_0^k \frac{1}{1-x^2}\,\mathrm{d}x = \frac{1}{2}\ln\left|\frac{p(k)}{q(k)}\right| + C$, where $0 < k < 1$ and $p(k)$ and $q(k)$ are expressions in terms of k, find $p(k)$ and $q(k)$. [3]

(c) Given that $\frac{x^2}{(1+x)(1+x^2)} = \frac{a}{1+x} + \frac{bx+c}{1+x^2}$, find the value of a, the value of b, and the value of c. [5]

(d) (i) Show that $I = \int \frac{x^2}{(1+x)(1+x^2)}\,\mathrm{d}x = \frac{1}{4}\ln\left[(1+x)^2(1+x^2)\right] - \frac{1}{2}\arctan(x) + C$ [5]

(ii) If $I = \frac{\pi}{4}$ when $x = 1$, find the value of C giving your answer in the form $m + n\ln r$ where $m, n, r \in \mathbb{R}$. [3]

(The exam continues on the next page)

2. [Maximum mark: 26]

In this question, you will investigate the value of expressions involving infinite nested square roots in the form $k(a)=\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\cdots}}}}}$ **where** $k(a)$ **is the exact value of** $\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\cdots}}}}}$ **and** a **is a positive integer.**

(a) Consider the expression $k(1)=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}$. Also, consider the following infinite sequence of terms u_n:

$$u_1=\sqrt{1+\sqrt{1}}\,,\ \ u_2=\sqrt{1+\sqrt{1+\sqrt{1}}}\,,\ \ u_3=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}\,,\ldots$$

Noting that $u_2=\sqrt{1+\left(\sqrt{1+\sqrt{1}}\right)}=\sqrt{1+(u_1)}=\sqrt{1+u_1}$, write down a formula for u_{n+1} in terms of u_n. [1]

(b) (i) Using your GDC, calculate the approximate value of each of the first ten terms in the sequence $u_1, u_2, u_3, \ldots$ Write the value of each term to an accuracy of five significant figures. [3]

(ii) Based on your results from (b) (i), deduce the value of $\lim\limits_{n\to\infty}(u_n-u_{n+1})$. [1]

(iii) $k(1)$ is the **exact** value of $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}$. Use your results from (a) and (b) (ii) to find an equation where $k(1)$ is the only variable. [4]

(iv) Show that $k(1)=\dfrac{1+\sqrt{5}}{2}$. [3]

(c) Hence, find the **exact** value of $k(2)=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}}$ [4]

(d) Find the **exact** value of $k(a)$ in terms of a. [4]

(e) (i) The value of $k(a)$ is an integer for some values of a. Find the first six values of a such that $k(a)$ is an integer. [3]

(ii) The values of a that generate an integer value for $k(a)$ form an infinite sequence with terms a_n, $n=1,2,3,\ldots$ Determine a formula for a_n in terms of n. [3]